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On a class of algebras defined by partitions

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Abstract

A class of associative (super) algebras is presented, which naturally generalize both the symmetric algebra $Sym(V)$ and the wedge algebra $\wedge(V)$, where V is a vector-space. These algebras are in a bijection with those subsets of the set of the partitions which are closed under inclusions of partitions. We study the rate of growth of these algebras, then characterize the case where these algebras satisfy polynomial identities.

1. Introduction

Throughout this paper let F denote a field of characteristic zero. Let V be a vector-space, $\dim V = k$, $T^n(V) = V \otimes \cdots \otimes V$ n-times, and $T(V)$ is the tensor algebra:

$$T(V) = \bigoplus_n T^n(V).$$

Both the Lie group $GL(V)$ (or the Lie algebra $gl(V)$) and the symmetric group S_n act naturally on $T^n(V)$, yielding the isotypic decomposition

$$T^n(V) = \bigoplus_{\lambda \in H(k,0;n)} W_\lambda, \tag{1}$$

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where $H(k,0;n) = \{\lambda = (\lambda_1, \lambda_2, \ldots) \mid \lambda_{k+1} = 0\}$. In fact, $W_\lambda \cong V_k^\lambda \otimes S^\lambda$, where S^λ is the Specht module (with character χ^λ), and V_k^λ is the corresponding $GL(V)$ (or $gl(V)$) irreducible module, which is unique—up to an isomorphism.

Recall the *symmetric algebra* $Sym(V)$ and the *wedge algebra* $\wedge(V)$:

$Sym(V) = T(V)/I_C$, where I_C is the two sided ideal in $T(V)$ generated by the elements $x \otimes y - y \otimes x$, $I_C = \langle x \otimes y - y \otimes x \mid x, y \in V\rangle$;
Similarly, $\wedge(V) = T(V)/I_E$ where $I_E = \langle x \otimes y + y \otimes x \mid x, y \in V\rangle$. We show in Example 4.1 that

$$I_C = \bigoplus_{\lambda;\ (1^2)\subseteq\lambda} W_\lambda \quad \text{and} \quad I_E = \bigoplus_{\lambda;\ (2)\subseteq\lambda} W_\lambda. \tag{2}$$

This leads to the following construction: Let $\mathbb{Y}(n) = Par(n)$ be the partitions of n and let $Par = \mathbb{Y} = \bigcup_n \mathbb{Y}(n)$ be all the partitions ($\mathbb{Y}$ is the so-called Young graph). A subset $\Omega \subseteq \mathbb{Y}$ which is closed under inclusions of partitions is called a *filter*. Given a subset $\Omega \subseteq \mathbb{Y}$, define

$$I_\Omega = \bigoplus_{\lambda\in\Omega} W_\lambda.$$

It is shown in Section 3 that the Littlewood–Richardson-rule (LR-rule) implies that Ω is a filter if and only if $I_\Omega = \bigoplus_{\lambda\in\Omega} W_\lambda$ is a two-sided ideal in $T(V)$. In such a case this yields the quotient algebra $A_\Omega = A_\Omega(V) = T(V)/I_\Omega$.

The above algebras have ‘super’ analogue algebras. In the ‘super’ case, $V = V_0 \oplus V_1$ with $\dim V_0 = k$ and $\dim V_1 = \ell$; again $T(V) = \bigoplus_n T^n(V)$ and S_n has a new sign-permutation action $*$ on $T^n(V)$; also the general-linear-Lie superalgebra $pl(V_0, V_1) = pl(k, \ell)$ acts on $T^n(V)$, see [3] for the details. This yields the following new isotypic decomposition:

$$T^n(V) = \bigoplus_{\lambda\in H(k,\ell;n)} W_\lambda, \tag{3}$$

where $H(k,\ell;n) = \{\lambda = (\lambda_1, \lambda_2, \ldots) \mid \lambda_{k+1} \leqslant \ell\}$, see Section 3 of [3]. Note the difference between W_λ in Eqs. (1) and (3): in (3) $W_\lambda \cong V_{k,\ell}^\lambda \otimes S^\lambda$, where $V_{k,\ell}^\lambda$ is the corresponding $pl(V)$ irreducible module, which is again unique—up to an isomorphism, see [3]. The corresponding construction of the algebras is unchanged: given a filter $\Omega \subseteq \mathbb{Y}$, again let $I_\Omega = \bigoplus_{\lambda\in\Omega} W_\lambda$; by Theorem 3.1, again I_Ω is a two sided ideal in $T(V)$, yielding the quotient algebra $A_\Omega = T(V)/I_\Omega$. In this paper we study some of the properties of these algebras, mostly when $\dim V < \infty$.

2. The main results

In the next section we study the correspondence between subsets $\Omega \subseteq \mathbb{Y}$ and subspaces $I_\Omega \subseteq T(V)$. Proposition 3.6 shows that such Ω is a filter (i.e., closed under inclusions of partitions) if and only if I_Ω is a two sided ideal in $T(V)$. In Section 9 it is proved that filters in $\mathbb{Y}$ are always finitely generated, see Theorem 9.2.

The filtration $T(V) = \bigoplus_n T^n(V)$ induces the filtration $A_\Omega(V) = A_\Omega = \bigoplus_n A_\Omega(n)$. By considering the dimensions $\dim A_\Omega(n)$ we can talk about the rate of growth of A_Ω. When the dimension of V is finite, it is shown in Section 5 that $A_\Omega(V)$ has an exponential growth—which is an integer:

Theorem 2.1 (see Theorem 5.2). *Let $V = V_0 \oplus V_1$ be finite dimensional and let $\Omega \subseteq \mathbb{Y}$ be a filter with the corresponding algebra $A_\Omega(V)$. Then $A_\Omega(V)$ has an exponential rate of growth α—which is an integer, and $0 \leqslant \alpha \leqslant \dim V$.*

When $\alpha = 1$, that rate of growth is polynomial. In Sections 6–8 we characterizes the algebras $A_\Omega(V)$ which are p.i., namely which satisfy polynomial identities. We prove

Theorem 2.2. *The algebra $A_\Omega(V)$ is p.i. if and only if it has a polynomial rate of growth.*

Section 6 treats the 'classical' case, namely the case $V_1 = 0$, so $V = V_0$. Theorem 6.1 gives necessary and sufficient conditions on Ω—for $A_\Omega(V)$ to be p.i., in which case $A_\Omega(V)$ always satisfies $[x, y]^r = 0$ for some r.

Sections 7 and 8 treat the 'super' case. Theorem 7.1 gives necessary and sufficient conditions on Ω—for A_Ω to be p.i., in which case $A_\Omega(V)$ always satisfies $h(x)^r = 0$ for some r. Here $h(x)$ is any polynomial identity of $E \otimes E$, and E is the infinite-dimensional Grassmann algebra. In this case it is possible to choose $h(x) = [x, y]^3$ hence, again, in the case of p.i., $A_\Omega(V)$ satisfies a power of the commutator $[x, y]$.

We remark that the Littlewood–Richardson rule (LR-rule) is applied, in a rather essential way, in the proofs of Proposition 3.6 and of Theorems 6.1 and 7.1.

In Section 4 we examine few special cases of such algebras A_Ω. For the classical symmetric algebra $Sym(V)$ we show that $Sym(V) \cong A_\Omega(V)$, where Ω is generated by the partition $(1,1) = (1^2)$. Similarly for the wedge algebra: $\wedge(V) \cong A_\Omega(V)$ where Ω is generated by the partition (2). It is well known that $Sym(V)$ is the associated graded algebra $gr(U(gl(V)))$ of the enveloping algebra $U(gl(V))$ of $gl(V)$. In the 'super'-case $V = V_0 \oplus V_1$, and Example 4.2 shows that the associated graded algebra $gr(U(pl(V_0 \oplus V_1)))$ of the enveloping algebra of the Lie superalgebra $pl(V_0, V_1)$ is also of the form $gr(U(pl(V_0 \oplus V_1))) \cong A_\Omega(V)$, where here $A_\Omega(V) \cong Sym(V_0) \otimes \wedge(V_1)$ (see for example [8]), and again Ω is generated by the partition $(1,1) = (1^2)$.

3. Filters in $\mathbb{Y}$ and ideals in $T(V)$

Let $V = V_0 \oplus V_1$, $\dim V_0 = k$ and $\dim V_1 = \ell$, and let $pl(V_0, V_1) = pl(k, \ell)$ denote the corresponding Lie superalgebra [8]. Notice that the 'classical' case is obtained by letting $V_1 = 0$: $pl(k, 0) = pl(V_0, 0) = gl(V_0)$.

Start with Eq. (1), let $\mu \vdash m$ and $\lambda \vdash r$ and in $T(V)$ consider $W_\mu W_\lambda \equiv W_\mu \otimes W_\lambda$; it is a $pl(k, \ell)$ module in a natural way, and we are interested in the $pl(k, \ell)$-module-decomposition of that module. The precise decomposition is given by the LR-rule, a rule which arises from the outer multiplication of characters of symmetric groups: Let $\chi^\mu \hat{\otimes} \chi^\lambda$ denote the outer-product of the characters χ^μ and χ^λ, then

$$\chi^\mu \hat{\otimes} \chi^\lambda = \sum_{\nu \vdash m+r} c^\nu_{\mu,\lambda} \chi^\nu,$$

where the coefficients $c^\nu_{\mu,\lambda}$ are given by the LR-rule, see for example [7]. In particular it follows from that rule that if $c^\nu_{\mu,\lambda} \neq 0$ then $\mu, \lambda \subseteq \nu$.

Theorem 3.1. *As $pl(k,\ell)$ modules,*

$$W_\mu W_\lambda \cong \left(V^\mu_{k,\ell} \otimes V^\lambda_{k,\ell}\right)^{\oplus (f^\mu f^\lambda)} \quad \textit{and} \quad V^\mu_{k,\ell} \otimes V^\lambda_{k,\ell} \cong \bigoplus_{\nu \in H(k,\ell;m+r)} \left(V^\nu_{k,\ell}\right)^{\oplus c^\nu_{\mu,\lambda}}.$$

In particular, if $V^\nu_{k,\ell}$ appears in $W_\lambda W_\mu$ then $\lambda, \mu \subseteq \nu$.

Proof. The first claim follows since, as $pl(k,\ell)$ modules, $W_\mu \cong (V^\mu_{k,\ell})^{\oplus f^\mu}$ and similarly for W_λ, see the remark after Eq. (1). We prove the second statement.

Let $M = V^\mu_{k,\ell} \otimes V^\lambda_{k,\ell}$, let $n = |\mu| + |\lambda|$ and let P be the matrix $P = \operatorname{diag}(x_1, \ldots, x_k, y_1, \ldots, y_\ell)$. Then

$$tr_M(P^{\otimes n}) = HS_\mu(x_1, \ldots, x_k, y_1, \ldots, y_\ell) HS_\lambda(x_1, \ldots, x_k, y_1, \ldots, y_\ell). \tag{4}$$

By the proof of Theorem 6.30 in [3] it suffices to show that

$$tr_M(P^{\otimes n}) = \sum_{\nu \in H(k,\ell;n)} c^\nu_{\mu,\lambda} HS_\nu(x_1, \ldots, x_k, y_1, \ldots, y_\ell). \tag{5}$$

By 5.1 of [6] the right-hand sides of Eqs. (4) and (5) are equal, which completes the proof. □

Definition 3.2. Recall that $\mathbb{Y} = \bigcup_n \mathbb{Y}(n)$ denote the set of all the partitions. Given a subset $\Omega \subseteq \mathbb{Y}$, define $I_\Omega \subseteq T(V)$ by

$$I_\Omega = \bigoplus_{\lambda \in \Omega} W_\lambda. \tag{6}$$

A subset $\Omega \subseteq \mathbb{Y}$ is called *a filter* if it is closed under inclusions of partitions: if $\mu \in \Omega$ and $\mu \subseteq \lambda$ then $\lambda \in \Omega$. Given partitions $\mu^1, \mu^2, \ldots \in \mathbb{Y}$, let $\Omega = \langle \mu^1, \mu^2, \ldots \rangle$ denote the filter generated by these partitions:

$$\left\langle \mu^1, \mu^2, \ldots \right\rangle = \left\{ \lambda \in \mathbb{Y} \mid \mu^i \subseteq \lambda \text{ for some } i \right\}.$$

Remark 3.3. Let $V = V_0 \oplus V_1$ with $\dim V_0 = k$ and $\dim V_1 = \ell$ finite. Let μ denote the $k+1 \times \ell+1$ rectangle: $\mu = ((\ell+1)^{k+1})$. By Eq. (3), if $\mu \subseteq \lambda$ then $W_\lambda = 0$. Given a filter Ω, let Ω_1 be the filter obtained by adding all $\mu \subseteq \lambda$ to Ω, then $I_\Omega = I_{\Omega_1}$. When $\dim V$ is finite, we shall therefore always assume that rectangle μ is in Ω.

A basic and obvious property of such a subspace $I_\Omega \subseteq T(V)$ is the following.

Proposition 3.4. *If $V^\nu_{k,\ell}$ appears in I_Ω then $W_\nu \subseteq I_\Omega$.*

The proof of Proposition 3.6 below requires the following lemma.

Lemma 3.5. *Let μ and ν be partitions such that $\mu \subseteq \nu$. Then there exists a partition λ such that χ^ν appears in $\chi^\mu \hat{\otimes} \chi^\lambda$. Moreover, if $\mu, \nu \in H(k,\ell)$ then also $\lambda \in H(k,\ell)$. Here $H(k,\ell) = \bigcup_n H(k,\ell;n)$.*

Proof. Let $a_1, \ldots, a_k$ be the lengths of the rows of ν/μ, then ν appears in $\chi^\mu \hat{\otimes} \chi^{(a_1)} \hat{\otimes} \cdots \hat{\otimes} \chi^{(a_k)}$. Therefore there is a χ^λ in $\chi^{(a_1)} \hat{\otimes} \cdots \hat{\otimes} \chi^{(a_k)}$ such that χ^ν appears in $\chi^\mu \hat{\otimes} \chi^\lambda$. The second statement follows from the last statement of Theorem 3.1. □

Proposition 3.6. *Let $\Omega \subseteq \mathbb{Y}$ be a subset with corresponding subspace $I_\Omega \subseteq T(V)$. Then Ω is a filter if and only if I_Ω is a two-sided ideal in $T(V)$.*

Proof. First, assume Ω is a filter and show that I_Ω is an ideal. It suffices to show the following: Let $\mu \in \Omega$ and let λ be any partition, then $W_\mu W_\lambda \subseteq I_\Omega$. This follows from the last statement of Theorem 3.1.

Next, assume I_Ω is an ideal and show that Ω is a filter. Let $\mu \in \Omega$, let $\mu \subseteq \nu$ and show $\nu \in \Omega$. Recall that $T(V) = \bigoplus_\lambda W_\lambda$. By definition, $W_\mu \subseteq I_\Omega$. By Lemma 3.5 there exist a partition λ such that χ^ν appears in $\chi^\mu \hat{\otimes} \chi^\lambda$ (i.e., $c^\nu_{\mu,\lambda} \neq 0$). It follows that $V^\nu_{k,\ell}$ appears in $W_\mu W_\lambda \subseteq I_\Omega W_\lambda \subseteq I_\Omega$. By Proposition 3.4 $W_\nu \subseteq I_\Omega$, and by the definition, $\nu \in \Omega$. □

The algebras $A_\Omega(V)$ can now be introduced.

Definition 3.7. Let $V = V_0 \oplus V_1$ with the corresponding W_λ's as in Eq. (3). Let $\Omega \subseteq \mathbb{Y}$ with the corresponding subspace $I_\Omega = I_\Omega(V)$ as in Eq. (6). Let $A_\Omega(V)$ be the quotient space:

$$A_\Omega(V) = T(V)/I_\Omega(V).$$

Clearly, $A_\Omega(V)$ can be identified with the subspace

$$A_\Omega(V) \equiv \bigoplus_{\lambda \notin \Omega} W_\lambda. \tag{7}$$

Remark 3.8. (1) In particular, the identification (7) implies that if $V \subseteq V'$ then $A_\Omega(V) \subseteq A_\Omega(V')$.

(2) When Ω is a filter, $I_\Omega(V)$ is a two-sided ideal and $A_\Omega(V)$ is an associative algebra.

These algebras $A_\Omega(V)$ are the subject of this paper.

4. Some examples

Few examples of algebras A_Ω are given below. We show that in the 'classical' case (namely $V = V_0, V_1 = 0$), both the symmetric algebra $Sym(V)$ and the wedge algebra $\wedge(V)$ are of the form $A_\Omega(V)$: $Sym(V) \cong A_{\langle(1^2)\rangle}(V)$ and $\wedge(V) \cong A_{\langle(2)\rangle}(V)$. In the general case, when $V = V_0 \oplus V_1$, $A_{\langle(1^2)\rangle} \cong Sym(V_0) \otimes \wedge(V_1)$. Note that the associated graded algebra $gr(U(pl(V_0 \oplus V_1)))$ of the enveloping algebra of the Lie superalgebra $pl(V_0, V_1)$ is also of that form: $gr(U(pl(V_0 \oplus V_1))) \cong Sym(V_0) \otimes \wedge(V_1)$ [8].

Example 4.1. Let $V = V_0$, $V_1 = 0$.

(1) Let $Sym(V)$ be the symmetric algebra of the vector space V. Then $Sym(V) \cong A_{\langle(1^2)\rangle}$, where $\Omega = \langle(1^2)\rangle = \{\lambda \mid (1^2) \subseteq \lambda\}$. Thus, as vector spaces, $Sym(V) \cong \bigoplus_n W_{(n)}$.
(2) Let $\wedge(V)$ be the wedge algebra of the vector space V. Then $\wedge(V) \cong A_{\langle(2)\rangle}$, where $\Omega = \langle(2)\rangle = \{\lambda \mid (2) \subseteq \lambda\}$. Thus, as vector spaces, $\wedge(V) \cong \bigoplus_n W_{(1^n)}$.

Proof. We prove part (2). The proof of part (1) is similar. Recall that $\wedge(V) = T(V)/I_E$ where $I_E = \langle x \otimes y + y \otimes x \mid x, y \in V\rangle$. We show that $I_E = \bigoplus_{\lambda;(2)\subseteq\lambda} W_\lambda$. Denote $I_E(n) = I_E \cap T^n(V)$. S_n acts on $T^n(V)$—hence on $I_E(n)$—(from the right, as in [3]) by permuting places, and we show first that it maps $I_E(n)$ into itself. $I_E(n)$ is spanned by elements of the form $x_1 \cdots x_{r-1}(x_r x_{r+1} + x_{r+1} x_r) x_{r+2} \cdots x_{r_n}$ (here $x_1 x_2 = x_1 \otimes x_2$, etc.). It suffices to verify for the transpositions $(i, i+1)$, and after some obvious reduction, to verify that $(1, 2)$ maps $x_1(x_2 v_3 + v_3 x_2)$ into I_E (and similarly, that $(2, 3)$ maps $(x_1 v_2 + v_2 x_2)x_3$ into I_E). This is clear, since

$$\begin{aligned}(1,2): x_1(x_2x_3 + x_3x_2) &\to x_2x_1x_3 + x_3x_1x_2 \\ &= (x_2x_1 + x_1x_2)x_3 - x_1(x_2x_3 + x_3x_2) + (x_1x_3 + x_3x_1)x_2 \in I_E.\end{aligned}$$

Let T_λ be a tableaux of shape λ. To T_λ corresponds the semi-idempotent $e_{T_\lambda} = R^+_{T_\lambda} C^-_{T_\lambda} \in FS_n$. Here $R_{T_\lambda}, C_{T_\lambda} \subseteq S_n$ are the subgroups of the T_λ-row and column permutations, with

$$R^+_{T_\lambda} = \sum_{p \in R_{T_\lambda}} p \quad \text{and} \quad C^-_{T_\lambda} = \sum_{q \in C_{T_\lambda}} \operatorname{sgn}(q) q.$$

We apply the following property of the W_λ's, which is well known: Let T_λ be *any* tableaux of shape λ, with the corresponding semi-idempotent e_{T_λ}, then $W_\lambda = T^n(V) e_{T_\lambda} FS_n$.

As usual, write $\lambda = (\lambda_1, \lambda_2, \ldots)$. We verify that $\bigoplus_{\lambda_1 \geqslant 2} W_\lambda \subseteq I_E$. Let $\lambda_1 \geqslant 2$ and let T_λ be a standard tableaux whose first row starts with 1 and 2. Write $e_{T_\lambda} = R^+_{T_\lambda} C^-_{T_\lambda}$, then $e_{T_\lambda} = (1 + (1,2)) \cdot a$ for some $a \in FS_n$. Given $\bar{x} = x_1 \cdots x_n \in T^n(V)$, we have $x_1 \cdots x_n(1 + (1,2)) = (x_1x_2 + x_2x_1)x_3 \cdots x_n \in I_E(n)$. Since $I_E(n)$ is closed under the S_n action, it follows that $\bar{x} e_{T_\lambda} FS_n = \bar{x}(1 + (1,2))aFS_n \subseteq I_E(n)$, which clearly implies that $W_\lambda \subseteq I_E$.

Next, verify that $I_E \subseteq \bigoplus_{\lambda_1 \geqslant 2} W_\lambda$. Let $\lambda = (2)$ with $T_{(2)}$ standard, then $e_{T_{(2)}} = 1 + (1,2)$. Given $x_1x_2 \in T^2(V)$, $x_1x_2e_{T_{(2)}} = x_1x_2 + x_2x_1 \in W_{(2)}$. By the LR-rule it follows that

$$x_1 \cdots x_{r-1}(x_r x_{r+1} + x_{r+1}x_r)x_{r+2} \cdots x_{r_n} \in \bigoplus_{\lambda;\lambda_1 \geqslant 2} W_\lambda.$$

Since these are the generators of I_E, the above inclusion follows. This completes the proof of the second example, and the proof of the first is similar. □

Next we consider the 'super' analogues of the previous examples.

Example 4.2. Let $V = V_0 \oplus V_1$, let $\Omega_1 = \langle (1,1) \rangle$ be the filter given by the partition $\mu = (1,1)$, and let $\Omega_2 = \langle (2) \rangle$ be the filter given by the partition $\nu = (2)$. We show that

$$A_{\langle (1,1) \rangle}(V_0, V_1) \cong \mathit{Sym}(V_0) \otimes \wedge(V_1),$$

and similarly

$$A_{\langle (2) \rangle}(V_0, V_1) \cong \mathit{Sym}(V_1) \otimes \wedge(V_0).$$

It follows from some basic facts in p.i. theory that both $A_{\langle (2) \rangle}(V_0, V_1)$ and $A_{\langle (1^2) \rangle}(V_0, V_1)$ satisfy the identity $[[x_1, x_2], x_3] = 0$.

Proof. Fix bases (either finite or infinite) $t_1, t_2, \ldots \in V_0$ and $u_1, u_2, \ldots \in V_1$. By abuse of notation, $t_i, u_j \in A_{\langle (1,1) \rangle}$. Also, $t_i \cdot t_j := t_i \otimes t_j + I_\Omega$ in $T(V)/I_\Omega$, and similarly for other products. These basis elements satisfy the following three commutation-relations in $A_{\langle (1,1) \rangle}$:

(1) $t_i t_j = t_j t_i$;
(2) $t_i u_j = u_j t_i$; and
(3) $u_i u_j = -u_j u_i$.

For example we prove (3). Note that here, the S_n action on $T^n(V)$ is the same as in [3], and is denoted by $*$. Now,

$$u_i \otimes u_j + u_j \otimes u_i = (u_i \otimes u_j) * e_{(1,1)} \in V^{\otimes 2} * e_{(1,1)} \subseteq W_{(1,1)} \subseteq I_\Omega,$$

hence $u_i u_j = -u_j u_i$ in A_Ω. Here $e_{(1,1)}$ is the semi-idempotent $e_{(1,1)} = 1 - (1,2)$. Similarly for the other two relations (1) and (2).

These commutation-relations imply the isomorphism $A_{\langle (1,1) \rangle} \cong \mathit{Sym}(V_0) \otimes \wedge(V_1)$. □

Similar arguments show that

$$A_{\langle(2)\rangle}(V_0, V_1) \cong Sym(V_1) \otimes \wedge(V_0).$$

We consider the classical case (namely $V_1 = 0$), and denote $\bigoplus_n W_{(n)} = A_C$ and $\bigoplus_n W_{(1^n)} = A_E$. As vector spaces, $Sym(V) \cong A_C$ and $\wedge(V) \cong A_E$, and these isomorphisms make A_C and A_E into algebras. Notice that if the sum $\bigoplus_n$ starts with $n = 0$ then these algebras have 1, while if it starts with $n = 1$, these algebras are without 1. For the next example we introduce the following notation:

$$A_C^* = \bigoplus_{n \geqslant 2} W_{(n)} \quad \text{and} \quad A_E^* = \bigoplus_{n \geqslant 2} W_{(1^n)}.$$

These are ideals in their respective algebras.

Example 4.3. Let $\Omega = \langle(2, 1)\rangle$, then we have the following algebra-isomorphisms:

$$A_{\langle(2,1)\rangle} \cong A_C \oplus A_E^* \quad \text{and also} \quad A_{\langle(2,1)\rangle} \cong A_C^* \oplus A_E. \tag{8}$$

This follows since, as vector spaces,

$$A_{\langle(2,1)\rangle} \cong \left(\bigoplus_n W_{(n)}\right) \oplus \left(\bigoplus_{n \geqslant 2} W_{(1^n)}\right) = A_C \oplus A_E^*,$$

and similarly

$$A_{\langle(2,1)\rangle} \cong \left(\bigoplus_{n \geqslant 2} W_{(n)}\right) \oplus \left(\bigoplus_n W_{(1^n)}\right) = A_C^* \oplus A_E.$$

We conclude with few more examples of ‘classical’ algebras A_Ω, namely, $V_1 = 0$, $V = V_0$.

Example 4.4. Let $\Omega = \langle(a)\rangle$ where $a > 0$. Recall that when $a = 2$, $A_{\langle(2)\rangle} \cong \wedge(V)$.

(1) $\dim V = k < \infty$. If $n > k(a - 1)$ then $T^n(V) \subseteq I_\Omega$. This follows since if $\lambda = (\lambda_1, \ldots, \lambda_k) \vdash n > k(a-1)$ then $\lambda_1 \geqslant a$. It follows that $\dim A_\Omega < \infty$. Thus, if $1 \notin A_\Omega$ then A_Ω is nilpotent: $(A_\Omega)^{k(a-1)+1} = 0$.
(2) $\dim V = \infty$. Assume $1 \notin A_\Omega$. By Remark 3.8(1) and by case 1 above, any finitely generated subalgebra is nilpotent. In particular, A_Ω is nil.

Remark 4.5. When $\Omega = \langle(2)\rangle$, $A_\Omega = A_{\langle(2)\rangle} \cong E$, where $E = E(V)$ is the corresponding Grassmann (Exterior) algebra. In particular, E is p.i., satisfying $[[x, y], z] = 0$ (even when $\dim V = \infty$). Note that $A_{\langle(1^3)\rangle}$ is not p.i. since it does not satisfy the condition $(b^2) \in \Omega$ of Theorem 6.1.

Question. (1) Let $\dim V_0 = \infty$, $\dim V_1 = 0$ and let $\Omega = \langle(3)\rangle$. Is the algebra $A_\Omega = A_{\langle(3)\rangle}$ a p.i. algebra? By Theorem 6.1 $A_{\langle(3)\rangle}$ is not p.i. if both $\dim V_0, \dim V_1 \neq 0$.

(2) Again let $\dim V_0 = \infty$, $\dim V_1 = 0$, and now $\Omega = \langle(2^2)\rangle$. Is $A_{\langle(2^2)\rangle}$ a p.i. algebra? Note that if $\dim V < \infty$ then $A_{\langle(2^2)\rangle}$ is p.i. by Theorem 6.1. Also note that if both $\dim V_0, \dim V_1 \neq 0$ then, again by Theorem 6.1, $A_{\langle(2^2)\rangle}$ is not p.i., since $(b, 1^{b-1}) \notin \langle(2^2)\rangle$ for all b's.

5. The growth of $\dim(A_\Omega(n))$

It is shown here that for any filter Ω, A_Ω has an exponential rate of growth and with an integer exponent. This is Theorem 5.2 below.

Definition 5.1. We say that the sequence d_n has exponential rate of growth $\geqslant \alpha$ if there exist a polynomial p such that for large enough n, $0 < p(n)$ and $d_n \geqslant p(n)\alpha^n$. In such a case we denote $\operatorname{Exp}(d_n) \geqslant \alpha$. Define $\operatorname{Exp}(d_n) \leqslant \alpha$ similarly, and define $\operatorname{Exp}(d_n) = \alpha$ if both conditions hold. Also denote $\operatorname{Exp}(d_n) = 0$ if $d_n = 0$ for all n large enough. Denote $\operatorname{Exp}(A_\Omega) = \operatorname{Exp}(d_n)$ where $d_n = \dim(A_\Omega(n))$.

Theorem 5.2. *Let $V = V_0 \oplus V_1$ be finite dimensional and let $\Omega \subseteq \mathbb{Y}$ be a filter with the corresponding algebra $A_\Omega = A_\Omega(V)$. Then A_Ω has an exponential rate of growth—which is an integer. More precisely, denote $d_n = \dim(A_\Omega(n))$, then there exists an integer $0 \leqslant \alpha \leqslant \dim(V_0) + \dim(V_1)$ such that $\operatorname{Exp}(A_\Omega) = \operatorname{Exp}(d_n) = \alpha$.*

Proof. The proof is given below, see Theorem 5.8. □

Definition 5.3. (1) Let $a_1, a_2, b \geqslant 0$ be integers such that $b \geqslant a_1, a_2$. Denote by $D(a_1, a_2, b)$ the following hook-rectangular diagram (i.e., partition)

$$D(a_1, a_2, b) = \left(b^{a_1}, a_2^{b-a_1}\right).$$

For example, $(7^3, 2^4) = D(3, 2, 7)$. Note that both the arm-length and the leg-length of $D(a_1, a_2, b)$ equal b. Also, $|(b^{a_1}, a_2^{b-a_1})| = (a_1 + a_2)b - a_1 a_2$.

(2) Let $\Omega \subseteq \mathbb{Y}$ be a filter and let $a \geqslant 0$ be an integer. We say that Ω satisfies the ath hook-rectangular condition if there exist a large enough integer $b > 0$ such that

$$D(a, 0, b), D(a-1, 1, b), \ldots, D(0, a, b) \in \Omega.$$

Remark 5.4. (1) If Ω satisfies the ath hook-rectangular condition (with b) then it also satisfies the $(a+1)$th hook-rectangular condition (with $b+1$). Thus, for any non-empty filter Ω, there exists $0 \leqslant a$ minimal such that Ω satisfies the ath—but not the $(a-1)$th—hook-rectangular condition; we denote it by $a = h_r = h_r(\Omega)$. See Remark 5.4(5) for an upper bound on $h_r(\Omega)$.

(2) By definition, Ω satisfies the 0th hook-rectangular condition exactly when $\Omega = \mathbb{Y}$, in which case $A_\Omega = 0$.

(3) Ω satisfies the 1th hook-rectangular condition if and only if $(b), (1^b) \in \Omega$ for some $b > 0$, hence if and only if $\dim A_\Omega$ is finite, in which case A_Ω is nilpotent if $1 \notin A_\Omega$.

(4) By Theorem 7.1, Ω satisfies the 2th hook-rectangular condition if and only if A_Ω is p.i.

(5) Let $V = V_0 \oplus V_1$ with $\dim V_0 = k$ and $\dim V_1 = \ell$, then by assumption Ω contains the $k+1 \times \ell+1$ rectangle μ^0: $\mu^0 = ((\ell+1)^{k+1}) \in \Omega$, see Remark 3.3. We show that $h_r(\Omega) \leqslant k + \ell + 1$. We show in Remark 5.9 that $h_r(\Omega) = k + \ell + 1$ exactly when $\Omega = \langle \mu^0 \rangle$. Check that Ω satisfies the $(k+\ell+1)$th hook-rectangular condition: If $a_1 + a_2 = k + \ell + 1$ then either $a_1 > k$ or $a_2 > \ell$, so for large enough b, $\mu^0 \subseteq D(a_1, a_2, b)$, hence $D(a_1, a_2, b) \in \Omega$. It follows that for any filter Ω, $\langle \mu^0 \rangle \subset \Omega \subseteq \mathbb{Y}$, there exists $0 \leqslant a \leqslant k + \ell + 1$ such that $a = h_r = h_r(\Omega)$.

Example 5.5. Let $V = V_0 \oplus V_1$ with $\dim V_0 = k$ and $\dim V_1 = \ell$. Let $\lambda \in H(k, \ell)$, $\lambda \neq \Phi$, and let $\Omega = \langle \lambda \rangle$. Denote

$$m(\lambda) := \max\{u + v \mid (u, v) \in \lambda\}.$$

It is not difficult to see that $a = h_r(\Omega) = m(\lambda) - 1$.

Lemma 5.6. *Given the integers $a_1, a_2 \geqslant 0$, assume $D(a_1, a_2, b) \notin \Omega$ for all integers $b \geqslant a_1, a_2$. Then* $\operatorname{Exp}(A_\Omega) \geqslant a_1 + a_2$.

Proof. Let $n = |(b^{a_1}, a_2^{b-a_1})|$. Since $D(a_1, a_2, b) \notin \Omega$, this implies that

$$\dim\bigl(A_\Omega(n)\bigr) \geqslant \dim W_{D(a_1,a_2,b)} \geqslant f^{D(a_1,a_2,b)}.$$

The proof now follows from the asymptotic estimates in Section 7 in [3], which show that for some polynomial p,

$$f^{D(a_1,a_2,b)} \geqslant p(n)(a_1 + a_2)^n$$

and $p(n) > 0$ if $n = |(b^{a_1}, a_2^{b-a_1})|$ is large enough. This completes the proof. □

The converse is given by the next lemma.

Lemma 5.7. *Let $a > 0$ be an integer, let Ω be a filter, and assume Ω satisfies the ath—hook-rectangular condition*: *there exists an integer $b > 0$ such that*

$$D(a, 0, b), D(a-1, 1, b), \ldots, D(0, a, b) \in \Omega.$$

Then $\operatorname{Exp}(A_\Omega) \leqslant a - 1$.

Proof. Here is a sketch of the proof.

Let $\lambda \vdash n$. If $\lambda \in \Omega$ then W_λ does not contribute to $\dim A_\Omega(n)$. Hence assume $\lambda \notin \Omega$. Let r_1 (respectively r_2) denote the number of rows (respectively columns) of

λ whose length is $\geqslant b$, then $D(r_1, r_2, b) \subseteq \lambda$. If $r_1 + r_2 \geqslant a$ then by assumption and by Remark 5.4(1) $D(r_1, r_2, b) \in \Omega$, hence also $\lambda \in \Omega$, a contradiction. Since $\lambda \notin \Omega$, $r_1 + r_2 \leqslant a - 1$. By the choice of r_1 and r_2 it follows that except for its $b \times b$ initial-corner-part, such λ is contained in the (r_1, r_2)-hook. Since $r_1 + r_2 \leqslant a - 1$, it follows that for some polynomial q, independent of λ, $f^\lambda \leqslant q(n)(r_1 + r_2)^n \leqslant q(n)(a-1)^n$ for all n: this follows by a slight extension of the asymptotic estimates in Section 7 of [3]. Recall that here $W_\lambda \cong V^\lambda_{k,\ell} \otimes S^\lambda$ (see Eq. (3)), hence $\dim W_\lambda = \dim(V^\lambda_{k,\ell}) \cdot f^\lambda$. By [3], $\dim(V^\lambda_{k,\ell})$ is polynomialy bounded, and also the total number of λ's in the above (r_1, r_2)-extended-hook, is polynomialy bounded—as a function of $|\lambda| = n$. It therefore follows that for some polynomial p,

$$\dim\big(A_\Omega(n)\big) \leqslant p(n)(a-1)^n.$$

This completes the proof. $\square$

We can now reformulate and prove Theorem 5.2.

Theorem 5.8. *Let $V = V_0 \oplus V_1$ be finite dimensional, with* $\dim V_0 = k$ *and* $\dim V_1 = \ell$. *Let $\Omega \subseteq \mathbb{Y}$ be a filter with the corresponding algebra A_Ω and let $a = h_r = h_r(\Omega)$ as in Remark* 5.4(1). *Assume $\Omega \neq \mathbb{Y}$, hence $a \geqslant 1$. Then* $\mathrm{Exp}(A_\Omega) = a - 1$.

Thus, in Example 5.5, $\mathrm{Exp}(A_\Omega) = m(\lambda) - 2$.

Proof. As was explained in Remark 5.4, such $a = h_r = h_r(\Omega)$ exists, and $a \geqslant 1$ since $\Omega \neq 0$. By definition there exist integers $a_1, a_2 \geqslant 0$ and $a_1 + a_2 = a - 1$, such that for all integers $b > a_1, a_2$, $D(a_1, a_2, b) \notin \Omega$. By Lemma 5.6 deduce that $\mathrm{Exp}(A_\Omega) \geqslant a - 1$. Conversely, Ω does satisfy the ath hook-rectangular condition, and by Lemma 5.7 $\mathrm{Exp}(A_\Omega) \leqslant a - 1$, which completes the proof. $\square$

Remark 5.9. Clearly, $\mathrm{Exp}(T(V)) = \dim V_0 + \dim V_1 = k + \ell$. The converse is also true: if $\mathrm{Exp}(A_\Omega) = k + \ell$ then $\Omega = \langle((\ell+1)^{k+1})\rangle$ and $A_\Omega = T(V)$. Indeed let $\Omega \subset \Omega_1$, a proper inclusion, and show that $\mathrm{Exp}(A_{\Omega_1}) \leqslant k + \ell - 1$. Let $\eta \in \Omega_1$, $\eta \notin \Omega$. Then $\eta \in H(k, \ell)$. Now $\eta \subseteq D(k, \ell, \eta_1 + \eta'_1)$, therefore $D(k, \ell, \eta_1 + \eta'_1) \in \Omega_1$. Consider λ's such that W_λ contribute to A_{Ω_1}, namely $\lambda \notin \Omega_1$. If $\lambda \notin \Omega_1$, it follows that either $\lambda_k \leqslant \eta_1 + \eta'_1$ or $\lambda'_\ell \leqslant \eta_1 + \eta'_1$. The asymptotics of f^λ for such λ's is $\leqslant (k + \ell - 1)^{|\lambda|}$; again, this follows by a slight extension of the asymptotic estimates in Section 7 of [3]. As in the previous lemma, this implies that $\mathrm{Exp}(A_{\Omega_1}) \leqslant k + \ell - 1$.

6. The 'classical' algebras A_Ω which are p.i.

Next we characterize those algebras A_Ω that are p.i., namely satisfy polynomial identities. Recall that $V = V_0 \oplus V_1$. In this section we consider the case $V_1 = 0$. Recall that $[x, y] = xy - yx$. We prove:

Theorem 6.1. *Let $V_1 = 0$, so $V = V_0$, and let* $\dim V = k < \infty$. *let Ω be a non-empty filter with the corresponding algebra $A_\Omega = T(V)/I_\Omega$. Then the following three conditions are equivalent.*

(1) *A_Ω is p.i.*
(2) *A_Ω has a polynomial rate of growth.*
(3) *There exist $c > 0$ such that $(c^2) \in \Omega$. In that case A_Ω satisfies the identity $[x_1, x_2] \cdots [x_{2\ell-1}, x_{2\ell}] = 0$ (hence also $[x, y]^\ell = 0$) where $\ell = c \cdot (\dim V - 1) + 1$.*

Proof. By Theorem 4.13 of [1], (1) implies (2).

Show that (2) implies (3): Assume $(c^2) \notin \Omega$ for all $c > 0$ and show that $\operatorname{Exp}(A_\Omega) \geqslant 2$, namely, that the growth of A_Ω is larger than polynomial. Indeed, as in the proof of Lemma 5.6, that assumption implies that

$$\dim A_\Omega(n) \geqslant \sum_{\lambda=(\lambda_1,\lambda_2)\vdash n} f^\lambda \simeq p(n) \cdot 2^n \tag{9}$$

for some polynomial $p(x)$, hence $\operatorname{Exp}(A_\Omega) \geqslant 2$.

Finally, show that (3) implies (1)—with the above polynomial identity. So, assume that $(c^2) \in \Omega$ and show that A_Ω satisfies the identity $[x_1, x_2] \cdots [x_{2\ell-1}, x_{2\ell}] = 0$, where $\ell = c \cdot (\dim V - 1) + 1$.

Let $s_1, s_2 \in T(V)$, then

$$[s_1, s_2] \in \bigoplus_{\lambda;|\lambda|-\lambda_1 \geqslant 1} W_\lambda. \tag{10}$$

Indeed, fix a basis $v_1, \ldots, v_k \in V$, and without loss of generality let $s_1 = v_{i_1} \cdots v_{i_p}$ $(= v_{i_1} \otimes \cdots \otimes v_{i_p})$ and $s_2 = v_{j_1} \cdots v_{j_q}$.

Now, $[v_i, v_j] \in W_{(1^2)}$ while, for example, $[v_{i_1} v_{i_2}, v_j] = [v_{i_1}, v_j] v_{i_2} + v_{i_1} [v_{i_2}, v_j] \in \bigoplus_{\lambda;|\lambda|-\lambda_1 \geqslant 1} W_\lambda$, etc. This verifies (10).

By the LR-rule, for $s_1, \ldots, s_{2\ell} \in T(V)$,

$$[s_1, s_2] \cdots [s_{2\ell-1}, s_{2\ell}] \in \bigoplus_{\lambda;|\lambda|-\lambda_1 \geqslant \ell} W_\lambda. \tag{11}$$

If λ has more than k parts then by assumption $\lambda \in \Omega$ hence $W_\lambda \subseteq I_\Omega$.

Assume now that $\lambda = (\lambda_1, \ldots, \lambda_k)$, with $|\lambda| - \lambda_1 \geqslant \ell$ and show that $\lambda_2 > c$. Assume not, then since $c \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_k$ therefore $|\lambda| - \lambda_1 = \lambda_2 + \cdots + \lambda_k \leqslant (k-1)c < \ell$, a contradiction (recall: $\ell = c \cdot (\dim V - 1) + 1$). It follows that in that case $\lambda_2 > c$ hence, by assumption, $\lambda \in \Omega$, so $W_\lambda \subseteq I_\lambda$, which is zero in A_Ω. Following Eq. (11) we see that $[s_1, s_2] \cdots [s_{2\ell-1}, s_{2\ell}] = 0$ in A_Ω. This proves part (2). □

7. The 'super' algebras A_Ω which are p.i.

Let $g(x_1,\dots,x_r)$ be a multilinear polynomial which is an identity of $E\otimes E$, where E is the infinite-dimensional Grassmann (Exterior) algebra. At the end of Section 8 we discuss such polynomials $g(x)$ of low degrees.

Here we prove

Theorem 7.1. *Let $V=V_0\oplus V_1$ where $\dim V_0,\dim V_1<\infty$, and let Ω be a filter. Then the following three conditions are equivalent.*

(1) *A_Ω is p.i.*
(2) *A_Ω has a polynomial rate of growth.*
(3) *$D(2,0,b), D(1,1,b), D(0,2,b)\in\Omega$ for some $b>0$. In that case A_Ω satisfies the identity*

$$q_{tr}(x)=g(x_1,\dots,x_r)g(x_{r+1},\dots,x_{2r})\cdots g(x_{(t-1)r+1},\dots,x_{tr})=0,$$

where $t=b^2$ and $g(x_1,\dots,x_r)$ is any multilinear identity of $E\otimes E$.

Proof. The proof is similar to, but more elaborate than that of Theorem 6.1.

As in the proof of Theorem 6.1, (1) implies (2).

Show that (2) implies (3): Assume $(a,1^b)\notin\Omega$ for all positive integers a,b. Repeating the analogue argument in the proof of Theorem 6.1, we conclude that

$$\dim A_\Omega(n)\geqslant\sum_{\lambda\in H(1,1;n)}f^\lambda=2^{n-1},\tag{12}$$

hence A_Ω is not p.i. Similarly if all $(b^2)\notin\Omega$ or if all $(2^b)\notin\Omega$. This shows that if A_Ω is p.i. then for some $b>0$, $D(2,0,b), D(1,1,b), D(0,2,b)\in\Omega$.

The proof that (3) implies (1)—and with that identity—is similar to, but more elaborate than the proof of the analogue part in Theorem 6.1. The proof follows from the following claims.

Lemma 7.2. *Denote $c(\lambda)=|\lambda|-\lambda_1=\lambda_2+\lambda_3+\cdots$.*

Write $T(V)=\bigoplus_\lambda W_\lambda$. Let $g(x_1,\dots,x_r)$ as above and let $m_1,\dots,m_r\in T(V)$, then

$$g(m_1,\dots,m_r)\in\bigoplus_{\lambda;\lambda_1,\lambda_1'\geqslant 2}W_\lambda=\bigoplus_{\lambda;c(\lambda),c(\lambda')\geqslant 1}W_\lambda.$$

The proof of this lemma is given in the next section, see Corollary 8.3.

Together with the LR-rule, Lemma 7.2 implies that for any $m_1,\dots,m_{rt}\in T(V)$,

$$q_{rt}(m)=g(m_1,\dots,m_r)g(m_{r+1},\dots,m_{2r})\cdots g(x_{(t-1)r+1},\dots,x_{tr})\in\bigoplus_{\lambda;c(\lambda),c(\lambda')\geqslant t}W_\lambda.$$

Lemma 7.3. *Let $t \geqslant b^2$ and let λ be a partition satisfying $c(\lambda), c(\lambda') \geqslant t$. Then either $(b, 1^{b-1}) \subseteq \lambda$ or $(b^2) \subseteq \lambda$ or $(2^b) \subseteq \lambda$.*

Proof. Assume $(b^2), (2^b), (b, 1^{b-1}) \not\subseteq \lambda$. Since $(b^2) \not\subseteq \lambda$, $\lambda_2 < b$. Similarly, $(2^b) \not\subseteq \lambda$ implies that $\lambda'_2 < b$, while $(b, 1^{b-1}) \not\subseteq \lambda$ implies that either $\lambda_1 < b$ or $\lambda'_1 < b$, say $\lambda'_1 < b$. In such a case, $\lambda_{b+1} = 0$ and deduce that

$$c(\lambda) = \lambda_2 + \cdots + \lambda_b \leqslant (b-1)^2 < b^2,$$

which is a contradiction. □

The proof that (3) implies (1) (in Theorem 7.1). This last lemma implies that

$$\bigoplus_{\lambda; c(\lambda), c(\lambda') \geqslant t} W_\lambda \subseteq \bigoplus_{\lambda \in \Omega} W_\lambda,$$

which, by Lemma 7.2 implies that $q_{rt}(x) = 0$ in A_Ω and therefore is an identity of A_Ω. This completes the proof of Theorem 7.1. □

8. A property of the polynomial identities of $E \otimes E$

Recall that

$$e_{(n)} = \sum_{\sigma \in S_n} \sigma \quad \text{and} \quad e_{(1^n)} = \sum_{\sigma \in S_n} (-1)^\sigma \sigma.$$

In this section we prove

Theorem 8.1. *Let $g(x_1, \ldots, x_d)$ be a multi-linear polynomial identity of $E \otimes E$, where E is the infinite-dimensional Grassmann (Exterior) algebra. Let $V = V_0 \oplus V_1$ and let $m_1, \ldots, m_d$ be monomials in $T(V)$ such that $m_1 \cdots m_d \in T^n(V)$. Also, recall that the group algebra FS_n has the super-action $*$ on $T^n(V)$* [3]. *Then*

$$g(m_1, \ldots, m_d) * e_{(1^n)} = g(m_1, \ldots, m_d) * e_{(n)} = 0.$$

By Corollary 8.3 *below, this implies that*

$$g(m_1, \ldots, m_d) \in \bigoplus_{\lambda; c(\lambda), (\lambda') \geqslant 1} W_\lambda.$$

The proof is given below.

The functions $f_I(\sigma)$. *First, recall the functions $f_I(\sigma)$ from Definition* 1.1 *of* [3]: *let $E = E_0 \oplus E_1$ be the usual decomposition of E, let $I \subseteq \{1, \ldots, d\}$ and let $a_1, \ldots, a_d \in E_0 \cup E_1$ such that $a_i \in E_1$ if and only if $i \in I$; then $f_I(\sigma) = \pm 1$ is given by the equation*

$$a_{\sigma(1)} \cdots a_{\sigma(d)} = f_I(\sigma) a_1 \cdots a_d. \tag{13}$$

The functions $f_I(\sigma)$ also appear naturally in the following context. Let $m_1, \ldots, m_d$ be monomials in $x_1, \ldots, x_n$ such that $m_1 \cdots m_d = x_1 \cdots x_n$. Given $\sigma \in S_d$, there is a unique permutation $\eta = \eta_\sigma \in S_n$ such that $m_{\sigma(1)} \cdots m_{\sigma(d)} = x_{\eta(1)} \cdots x_{\eta(n)}$. Let $I = \{1 \leqslant i \leqslant d \mid \deg m_i \text{ is odd}\}$. Then

$$\operatorname{sgn}(\eta) = \operatorname{sgn}(\eta_\sigma) = f_I(\sigma). \tag{14}$$

Lemma 8.2. *Let*

$$p(x_1, \ldots, x_d) = \sum_{\sigma \in S_d} \alpha_\sigma x_{\sigma(1)} \cdots x_{\sigma(d)}.$$

Then $p(x_1, \ldots, x_d)$ is a polynomial identity of $E \otimes E$ if and only if for any pair of subsets $I_1, I_2 \subseteq \{1, \ldots, d\}$,

$$\sum_{\sigma \in S_d} \alpha_\sigma f_{I_1}(\sigma) f_{I_2}(\sigma) = 0.$$

Proof. The proof follows straightforward from Eq. (13), since

$$a_{\sigma(1)} \cdots a_{\sigma(d)} \otimes b_{\sigma(1)} \cdots b_{\sigma(d)} = f_{I_1}(\sigma) f_{I_2}(\sigma)(a_1 \cdots a_d \otimes b_1 \cdots b_d). \qquad \square$$

The proof of Theorem 8.1. Let $g(x_1, \ldots, x_d)$ be a multi-linear polynomial identity of $E \otimes E$ and write

$$g(x_1, \ldots, x_d) = \sum_{\sigma \in S_d} \alpha_\sigma x_{\sigma(1)} \cdots x_{\sigma(d)}. \tag{15}$$

Let $m_1, \ldots, m_d$ be monomials in $T(V)$ such that $m_1 \cdots m_d \in T^n(V)$, namely, $m_1 \cdots m_d = z_1 \cdots z_n$ with $z_1, \ldots, z_n \in V$. By embedding V in a large enough vector-space we may assume w.l.o.g. that $z_1, \ldots, z_n$ are linearly independent, as well as super-homogeneous, namely $z_1, \ldots, z_n \in V_0 \cup V_1$. If $m_i = z_{i_1} \cdots z_{i_r}$, then m_i is super-homogeneous of degree $\delta(m_i) = 1$ if the number of z_{i_j}'s which are in V_1 is odd; otherwise $\delta(m_i) = 0$. It follows that $m_1, \ldots, m_d$ are also super-homogeneous, and we let I_1 be the indices i with m_i having super-degree 1: $I_1 = \{1 \leqslant i \leqslant d \mid \delta(m_i) = 1\}$. As a monomial, each m_i has a degree, and we let I_2 denote the i's with m_i of odd degree: $I_2 = \{1 \leqslant i \leqslant d \mid \deg(m_i) \text{ is odd}\}$. For example, let $x \in V_0$, $y \in V_1$, and let $m = xyyxy$, then $\delta(m) = 1$ and $\deg(m) = 5$.

For each $\sigma \in S_d$ let $\eta = \eta_\sigma$ be the unique permutation in S_n such that $m_{\sigma(1)} \cdots m_{\sigma(d)} = z_{\eta(1)} \cdots z_{\eta(n)}$ (see Eq. (14)). It follows from the definition of $f_I(\sigma)$ (namely, from Eq. (13)) that for $\sigma \in S_d$,

$$m_{\sigma(1)} \cdots m_{\sigma(d)} = f_{I_1}(\sigma) \cdot (m_1 \cdots m_d) * \eta_\sigma = (m_1 \cdots m_d) * \big(f_{I_1}(\sigma)\eta_\sigma\big). \tag{16}$$

By Eq. (14)

$$\operatorname{sgn}(\eta)=\operatorname{sgn}(\eta_\sigma)=f_{I_2}(\sigma). \tag{17}$$

By Eqs. (15) and (16)

$$g(m_1,\dots,m_d)=(m_1\cdots m_d)*\left(\sum_{\sigma\in S_d}\alpha_\sigma f_{I_1}(\sigma)\eta_\sigma\right). \tag{18}$$

To calculate $g(m_1,\dots,m_d)*e_{(1^n)}$, note that $e_{(1^n)}=\sum_{\theta\in S_n}\operatorname{sgn}(\theta)\theta$. Since $(m*\tau)*\pi=m*(\tau*\pi)$ (see Lemma 1.5 of [3]), we have

$$g(m_1,\dots,m_d)*e_{(1^n)}=(m_1\cdots m_d)*\left(\sum_{\sigma\in S_d}\alpha_\sigma f_{I_1}(\sigma)\eta_\sigma\sum_{\theta\in S_n}\operatorname{sgn}(\theta)\theta\right). \tag{19}$$

But by Eq. (17)

$$\eta_\sigma\sum_{\theta\in S_n}\operatorname{sgn}(\theta)\theta=\operatorname{sgn}(\eta_\sigma)\sum_{\theta\in S_n}\operatorname{sgn}(\theta)\theta=f_{I_2}(\sigma)\sum_{\theta\in S_n}\operatorname{sgn}(\theta)\theta, \tag{20}$$

hence

$$g(m_1,\dots,m_d)*e_{(1^n)}=(m_1\cdots m_d)*\left(\sum_{\sigma\in S_d}\alpha_\sigma f_{I_1}(\sigma)f_{I_2}(\sigma)\sum_{\theta\in S_n}\operatorname{sgn}(\theta)\theta\right) \tag{21}$$

which equals zero by Lemma 8.2. This shows that $g(m_1,\dots,m_d)*e_{(1^n)}=0$. The proof that $g(m_1,\dots,m_d)*e_{(n)}=0$ is essentially the same, but with I_2 empty, i.e., $f_{I_2}(\sigma)=1$ for all $\sigma\in S_d$. □

Corollary 8.3. *Let $g(x_1,\dots,x_r)$ be a multilinear polynomial which is an identity of $E\otimes E$, where E is the infinite-dimensional Grassmann algebra. Let $m_1,\dots,m_r\in T(V)$ such that $m_1\cdots m_r\in T^n(V)$, then*

$$g(m_1,\dots,m_r)\in\bigoplus_{\lambda\vdash n;\,\lambda_1,\lambda_1'\geqslant 2}W_\lambda.$$

Proof. We need to show that the components of $g(m_1,\dots,m_r)$ in both $W_{(n)}$ and in $W_{(1^n)}$ are zero. By [3], $W_{(n)}=(T^n(V))*e_{(n)}$, and similarly $W_{(1^n)}=(T^n(V))*e_{(1^n)}$. Thus, $g(m_1,\dots,m_r)$ has a component $\neq 0$ in $W_{(n)}$ if and only if $g(m_1,\dots,m_r)*e_{(n)}\neq 0$, and similarly for $g(m_1,\dots,m_r)*e_{(1^n)}$. This implies the proof. □

Remark 8.4. We conclude this section with two remarks about the identities of $E\otimes E$.

(1) A. Popov [5] showed that $E\otimes E$ satisfies the following two identities: $[[x,y]^2,x]=0$ and $[[[x_1,x_2],[x_3,x_4]],x_5]=0$. Thus, in Theorem 8.1 we can choose $g(x)$ to be either the multilinearization of $[[x,y]^2,x]$ (which is of degree 5) or the polynomial

$[[[x_1, x_2], [x_3, x_4]], x_5]$. Moreover, Popov also showed that these identities are of minimal degrees (i.e., $E \otimes E$ satisfies no identity of degree four), and the above two identities generate all the identities of $E \otimes E$.

(2) Explicit identities of $E \otimes E$ can also be obtained via cocharacters. The cocharacters of E are contained in the $(1,1)$ hook, [4]. Hence, by [2], the cocharacters of $E \otimes E$ are contained in the $(2,2)$ hook. It follows that any element of the two sided ideal $I_{(3,3,3)} \subseteq FS_9$, when realized as a polynomial, is an identity of $E \otimes E$. This allows the construction of explicit such identities—of degree 9. For example, $s_3^3[x_1, x_2, x_3] = 0$ is such an identity, where $s_3[x_1, x_2, x_3]$ is the standard polynomial of degree 3. Since $s_3[x_1, x_2, 1] = [x_1, x_2]$, it follows that $E \otimes E$ satisfies $[x_1, x_2]^3 = 0$, and in Theorem 8.1 we can choose $g(x_1, \ldots, x_d) = g(x_1, \ldots, x_6)$ to be the multilinearization of $[x_1, x_2]^3$.

9. Filters in $\mathbb{Y}$ are finitely generated

The following is obvious: let $\Omega_1, \Omega_2 \subseteq \mathbb{Y}$ be two filters, then $\Omega_1 \cup \Omega_2$ is a filter, and $I_{\Omega_1 \cup \Omega_2} = I_{\Omega_1} + I_{\Omega_2}$. Similarly for more filters. For example let μ be a partition and let $\Omega = \{\lambda \in \mathbb{Y} \mid \mu \subseteq \lambda\}$, then Ω is a filter, which we denote by $\langle \mu \rangle$. Similarly $\langle \mu, \eta \rangle = \{\lambda \in \mathbb{Y} \mid \mu \subseteq \lambda \text{ or } \eta \subseteq \lambda\}$ is a filter, and $\langle \mu, \eta \rangle = \langle \mu \rangle \cup \langle \eta \rangle$, etc.

We prove below that every filter $\Omega \subseteq \mathbb{Y}$ is finitely generated, i.e., there exist r and partitions $\mu^1, \ldots, \mu^r$ such that $\Omega = \langle \mu^1, \ldots, \mu^r \rangle$. By standard arguments, this is equivalent to proving a.c.c. (ascending chain condition) on filters.

Definition 9.1. Recall the notation $H(k, \ell; n)$ for the partitions of n in the (k, ℓ) hook:

$$H(k, \ell; n) = \bigl\{\lambda = (\lambda_1, \lambda_2, \ldots) \bigm| \lambda_{k+1} \leqslant \ell\bigr\} \quad \text{and} \quad H(k, \ell) = \bigcup_n H(k, \ell; n).$$

Thus $\mathbb{Y}_k = H(k, 0)$ is the "strip" of the partitions with at most k parts. Clearly, $\mathbb{Y}_k$ is the complement of the filter $\Omega = \langle (1^{k+1}) \rangle$. Therefore a filter that contains (1^{k+1}) is called 'a filter in $\mathbb{Y}_k$.' Similarly, a filter in the hook $H(k, \ell)$ is any filter that contains the $(k+1) \times (\ell+1)$ rectangle, i.e., the partition $((k+1)^{\ell+1})$.

Theorem 9.2. *Any filter in $\Omega \subseteq \mathbb{Y}$ is finitely generated.*

Proof. Let $\mu \in \Omega$, then μ is contained by some $(k+1) \times (\ell+1)$ rectangle. It follows that the complement of Ω is contained by the (k, ℓ)-hook $H(k, \ell)$. We prove the theorem under the assumption that Ω is a filter in $\mathbb{Y}_k$, namely the complement of Ω is contained in the 'strip' $\mathbb{Y}_k = H(k, 0)$. The proof of the general (i.e., 'hook')-case is similar.

The proof of the k-strip case is by induction on $k \geqslant 1$.

The case $k = 1$ is obvious: The complement of Ω is contained in

$$\mathbb{Y}_1 = \bigl\{(n) \bigm| n = 1, 2, \ldots\bigr\}.$$

Let $\Omega = \Omega_1 \subset \Omega_2$, then $(n) \in \Omega_2$ for n large enough. Since Ω_2 is a filter, it follows that its complement is a subset of the finite set $\{(1), (2), \ldots, (n-1)\}$, and the proof follows.

Next, assume that the complement of Ω is contained in $\mathbb{Y}_k$ (namely $(1^{k+1}) \in \Omega$), and that the theorem is true for filters whose complements are contained by the strips $\mathbb{Y}_r$ when $r < k$.

Assume $\Omega = \Omega_1 \subset \Omega_2$, then Ω_2 contains a partition μ with at most k parts: $\mu = (a_1, \ldots, a_k)$. Denote $\Omega_2' = \langle \mu, (1^{k+1}) \rangle$, so $\Omega_2' \subseteq \Omega_2$. It is therefore suffices to prove a.c.c. on chains of filters $\Omega_2' \subseteq \Omega_3 \subseteq \Omega_4 \subseteq \cdots$, i.e., that start with Ω_2'. Hence we consider the complement of Ω_2'.

Let $B = B(\mu)$ denote the following finite-union of sets of partitions

$$B(\mu) = \bigcup_{r=0}^{k-1} \left(\bigcup_{0 \leqslant b_k \leqslant \cdots \leqslant b_{r+1} \leqslant a_{r+1}-1} \bigl\{ (x_1, \ldots, x_r, b_{r+1}, \ldots, b_k) \bigm| x_1 \geqslant \cdots \geqslant x_k \geqslant b_{r+1} \bigr\} \right)$$

(when $r = 0$ the corresponding set $\{(b_1, \ldots, b_k) \mid 0 \leqslant b_k \leqslant \cdots \leqslant b_1 \leqslant a_1 - 1\}$ is finite). Let $\lambda \in \mathbb{Y}_k$. It is not difficult to see that $\lambda \notin B(\mu)$ iff $\mu \subseteq \lambda$, namely $a_i \leqslant \lambda_i$ for all i, and since Ω_2' is a filter, iff $\lambda \in \Omega_2'$. Thus, if $\lambda \notin \Omega_2'$ then $\lambda \in B(\mu)$, so for some $0 \leqslant r \leqslant k-1$ and some $0 \leqslant b_k \leqslant \cdots \leqslant b_{r+1} \leqslant a_{r+1} - 1$,

$$\lambda \in \bigl\{ (x_1, \ldots, x_r, b_{r+1}, \ldots, b_k) \bigm| x_1 \geqslant \cdots \geqslant x_r \geqslant b_{r+1} \bigr\}.$$

Note that $\{(x_1, \ldots, x_r, b_{r+1}, \ldots, b_k) \mid x_1 \geqslant \cdots \geqslant x_r \geqslant b_{r+1}\}$ is isomorphic to $\mathbb{Y}_r$ under the correspondence

$$(x_1, \ldots, x_r, b_{r+1}, \ldots, b_k) \leftrightarrow (x_1 - b_{r+1}, \ldots, x_r - b_{r+1}),$$

and this isomorphism preserves inclusions of partitions. Also, here $r < k$.

By induction, each such set satisfies a.c.c. (for filters), hence the above finite union $B(\mu)$ also satisfies that condition, and the proof of the theorem follows. $\square$

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